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The Hausdorff Dimension of Random Walks and the Correlation Length Critical Exponent in Euclidean Field Theory

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We study the random walk representation of the two-point function in statistical mechanics models near the critical point. Using standard scaling arguments, we show that the critical exponent ν describing the vanishing of the physical mass at the critical point is equal to ν_θ/d_w, where d_w is the Hausdorff dimension of the walk, and ν_θ is the exponent describing the vanishing of the energy per unit length of the walk at the critical point. For the case of $O(N)$ models, we show that $\nu_\theta = \varphi$, where φ is the crossover exponent known in the context of field theory. This implies that the Hausdorff dimension of the walk is φ/ν for $O(N)$ models.

KEY WORDS: Random walks in field theory; Hausdorff dimension of random walks; correlation length exponent.

1. INTRODUCTION

The two-point function is a quantity of central interest in statistical mechanics models including lattice-regulated quantum field theories. Let $G(r, t)$ be the two-point function between 0 and r in a model with one parameter t. Let $t=0$ be a critical point. Near $t=0$, the two-point function in three dimensions has a scaling form of the type (ref. 1, I-1-2)

$$G(r, t) = \frac{1}{r^{1+\eta}} g(rt^\nu) \tag{1}$$

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The scaling function $g(x)$ decays exponentially for large x, so that

$$m_p \sim t^\nu \tag{2}$$

is the physical mass. For free field theory, $\nu = 1/2$, and a departure from that typically signifies nontrivial interactions in the underlying field theory.

Many techniques exist for the calculation of ν. These calculations usually do not facilitate an understanding of why ν in an interacting theory deviates from 1/2. In this paper, we present a physical understanding of the exponent ν based on random walks. The two-point function in any statistical mechanics model admits a random walk representation.[2] This representation is obtained by writing the two-point function as a sum of an infinite number of terms with a one-to-one correspondence between each term and a random walk between the two points. The random walk representation provides a useful picture of the two-point function. The random walk describes the propagation from one point to the other and this walk takes place in the presence of background loops which represent vacuum fluctuations. In an interacting theory, pieces of the walk interact with each other as well as with background loops. Both these effects are important and combine to give a ν different from 1/2 for the interacting theory. We study this particular aspect of the random walk representation of the two-point function in some detail here.

In Section 2, we show that the random walk representation leads to an expression for the two-point function $G(r, t)$ of the form

$$G(r, t) = \int_0^\infty dl\, S(l, t)\, P(l, r, t) \tag{3}$$

$S(l, t)$ is the energy factor, and $P(l, r, t)$ is the entropy factor. Both of these include the effects of interactions. $P(l, r, t)$ is the probability density of walks of fixed length l:

$$P(l, r, t) \geqslant 0, \quad \forall l, r, t; \qquad \int d^3r\, P(l, r, t) = 1, \quad \forall l, t \tag{4}$$

Just as the two-point function has a scaling form near $t = 0$, we assume that both $S(l, t)$ and $P(l, r, t)$ have scaling forms near $t = 0$. We will be interested in the Hausdorff dimension d_w of the walk near $t = 0$.

The mean distance $R(l)$ of a walk of length l is proportional to the square root of the second moment of the probability density with respect to r. For $t \to 0$, the probability distribution of walks will be spread out, and $R(l)$ will diverge for large l with a leading behavior of the type

$$R(l) \sim l^{\nu_w} \tag{5}$$

$d_w = 1/\nu_w$ is the Hausdorff dimension of the walk.

The energy factor $S(l, t)$ is the weighted sum of all walks of length l. Therefore, $\theta(t) \equiv -(1/l) \ln S(l, t)$ for $l \to \infty$ is an appropriate definition of the energy per unit length of the walk. Near $t=0$ walks of all lengths become equally important, which implies that $\theta(t)$ vanishes as $t \to 0$. We assume the leading behavior

$$\theta(t) \sim t^{\nu_\theta} \tag{6}$$

ν_θ is another quantity of interest to us. Both (5) and (6) are similar in spirit to (2) and follow from the scaling form we assume for $P(l, r, t)$ and $S(l, t)$ just as (2) follows from (1).

The scaling forms for $S(l, t)$ and $P(l, r, t)$ when combined in (3) must be consistent with (1). Using this, we show in Section 2 that

$$\nu = \nu_w \nu_\theta \tag{7}$$

The content of (7) is that the nonanalyticity in (2) near $t=0$ has two sources: (a) a nonanalyticity in $\theta(t)$, which controls the length l, and (b) a nonanalyticity in the relation $R(l)$ between the mean distance and the length. If the underlying theory in the vicinity of the critical point is free, then $\nu_\theta = 1$ and $\nu_w = 1/2$.

We consider $O(N)$ models in Section 3 and show that for these models,

$$\nu_\theta = \varphi \tag{8}$$

where φ is a crossover exponent (ref. 1, II-5-3). Substitution of (8) in (7) gives

$$d_w = \frac{1}{\nu_w} = \frac{\varphi}{\nu} \tag{9}$$

2. DERIVATION OF THE SCALING LAW $\nu = \nu_w \nu_\theta$

The two-point function $G(r, t)$ can always be cast in the form[2]

$$G(r, t) = \sum_{w:0 \to r} \Omega(w, t) \tag{10}$$

The sum is over all random walks w connecting 0 and r, and (10) is called the random walk representation of the two-point function. $\Omega(w, t)$ depends upon the model and is always positive. By grouping together all walks of length l, (10) can be rewritten as[4]

$$G(r, t) = \int_0^\infty dl\, \tilde{\Omega}(l, r, t) \tag{11}$$

Define

$$S(l, t) = \int d^3r\, \tilde{\Omega}(l, r, t) \tag{12}$$

and

$$P(l, r, t) = \frac{\tilde{\Omega}(l, r, t)}{S(l, t)} \tag{13}$$

Substitution of (13) in (11) gives (3) with (13) satisfying the property (4).

In a manner similar to (1), both $S(l, t)$ and $P(l, r, t)$ are expected to have scaling forms. For $S(l, t)$, we assume the following general form:

$$S(l, t) = \frac{1}{l^{\eta_p}}\, s(lt^{\nu_\theta}) \tag{14}$$

with an exponential decay for s at large argument. This is identical to the form for $G(r, t)$ in (1). From (14), it is clear that the energy per unit length $\theta(t)$ is given by (6).

Next, we have to write down the scaling form for $P(l, r, t)$. Here we have to keep in mind that it has to be in concordance with (4), and with the combination of (1), (3), and (14). This results in the following general form for $P(l, r, t)$:

$$P(l, r, t) = \frac{1}{r^3}\, p(rl^{-\nu_w}, rt^\nu) \tag{15}$$

The prefactor $1/r^3$ is in agreement with (4). The leading behavior of the mean distance of the walk for large l is given by (5). The combination rt^ν in (15) is in accordance with (1) and (3).

Further, consistency of (14), (15), and (3) with (1) gives the scaling law (7). It also results in a scaling relation for the anomalous dimension η_p,

$$\eta_p = 1 - (2 - \eta)\nu_w \tag{16}$$

[4] In the following equation, the integral is to be understood as a limit of a sum.

The scaling relations (7) and (16) involve three exponents defined in the walk picture: ν_w, ν_θ, and η_p. In the next section, we focus on the $O(N)$ models to obtain (8). Equations (8), (7), and (16) are three relations for those three exponents.

3. $O(N)$ MODELS AND ν_θ

At fixed spatial cutoff $a = 1/\Lambda$, the $O(N)$ symmetric Lagrangian is

$$\mathscr{L}(\boldsymbol{\phi}) = \frac{1}{2}(\partial\boldsymbol{\phi}\cdot\partial\boldsymbol{\phi}) + \frac{1}{2}(m_c^2 + t)(\boldsymbol{\phi}\cdot\boldsymbol{\phi}) + \frac{\lambda_c}{4!}(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2 \tag{17}$$

m_c and λ_c are chosen so that $t = 0$ is the critical point.

The first step in deriving the random walk representation for the two-point function is to write the interaction term as

$$\exp\left[-\frac{\lambda_c}{4!}(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2\right] = \frac{1}{(2\pi)^{1/2}}\int d\sigma \exp\left[-\frac{1}{2}\sigma^2 + b\sigma(\boldsymbol{\phi}\cdot\boldsymbol{\phi})\right] \tag{18}$$

with

$$b = i\left(\frac{2\lambda_c}{4!}\right)^{1/2} \tag{19}$$

The partition becomes

$$\begin{aligned} Z(t) &= \int [d\boldsymbol{\phi}][d\sigma] \exp\left\{-\frac{1}{2}\int d^3x\, [\sigma^2 + \boldsymbol{\phi}\cdot(H+t)\boldsymbol{\phi}]\right\} \\ &= \int [d\sigma] \exp\left\{-\frac{N}{2}\operatorname{Tr}\ln(H+t) - \frac{1}{2}\int d^3x\sigma^2\right\} \end{aligned} \tag{20}$$

with

$$H = -\partial^2 + m_c^2 - 2b\sigma \tag{21}$$

The two-point function is

$$\begin{aligned} G(r,t)\delta^{ij} &= \langle \phi^i(0)\phi^j(r)\rangle \\ &= \int_0^\infty dl \exp(-tl)\frac{1}{Z(t)}\int [d\sigma] \\ &\quad \times \exp\left\{-\frac{N}{2}\operatorname{Tr}\ln(H+t) - \frac{1}{2}\int d^3x\sigma^2\right\}\Psi(l,r) \end{aligned} \tag{22}$$

In (22), $\Psi(l, r)$ is the solution to

$$-\frac{\partial}{\partial l}\Psi(l, r) = H\Psi(l, r) \tag{23}$$

with the initial condition

$$\Psi(0, r) = \delta^3(r) \tag{24}$$

$\Psi(l, r)$ is the propagation kernel for the Hamiltonian H. It has a standard path integral representation in which the sum is over all paths from the origin to r in proper time l. The length L of these paths is proportional to l and inversely proportional to the spatial cufoff a.

Therefore the integrand in (22) is indeed $\tilde{\Omega}(l, r, t)$ as defined in (11).[5] Hence $S(l, t)$ defined in (12) can be written as

$$S(l, t) = e^{-tl} S_I(l, t) \tag{25}$$

where

$$S_I(l, t) = \frac{1}{Z(t)} \int [d\sigma] \exp\left\{-\frac{N}{2}\operatorname{Tr}\ln(H+t) - \frac{1}{2}\int d^3x\, \sigma^2\right\} \times \left[\int d^3r\, \Psi(l, r)\right] \tag{26}$$

$S_I(l, t)$ incorporates the interaction of the walk in $\Psi(l, r)$ with the background loops in $\operatorname{Tr}\ln(H+t)$. If there is no interaction term in (17) ($\lambda_c = 0$), then $S_I(l, t) = 1$, $\eta_p = 0$, and $v_\theta = 1$.

The key issue is to understand the behavior of $S_I(l, t)$ when $N \geqslant 1$. In particular, the energy per unit length of the walk is

$$\theta(t) = t - \lim_{l \to \infty} \frac{\ln(S_I(l, t))}{l} \tag{27}$$

The second term on the right-hand side of (27) is the contribution from the interaction of the walk with the background loops. We would like to know its behavior near $t = 0$. We will now show that for small t,

$$\lim_{l \to \infty} \frac{\ln(S_I(l, t))}{l} = t - ct^{\varphi} + \text{higher powers in } t \tag{28}$$

This will result in (6) and (8).

[5] To make the explicit connection to (10), one just has to write down the path integral sum of $\Psi(l, r)$.

Toward this end, we softly break the $O(N)$ symmetry in (17) to an $O(K) \times O(N-K)$ symmetry by introducing a different mass term for the K-component $\boldsymbol{\phi}_1$ field and the $(N-K)$-component $\boldsymbol{\phi}_2$ field[6]:

$$(m_c^2 + t)(\boldsymbol{\phi} \cdot \boldsymbol{\phi}) \rightarrow (m_c^2 + t)(\boldsymbol{\phi}_1 \cdot \boldsymbol{\phi}_1) + (m_c^2 + t')(\boldsymbol{\phi}_2 \cdot \boldsymbol{\phi}_2) \tag{29}$$

The point $t = t' = 0$, where the $O(N)$ symmetry is broken, is the critical point of interest to us. But for the model with the asymmetric term (29), there are two critical lines in the (t, t') plane (ref. 1, II-5-3). One critical line corresponds to the breaking of the $O(K)$ symmetry, and the other line corresponds to the breaking of the $O(N-K)$ symmetry. These lines meet at the bicritical point $t = t' = 0$ where the $O(N)$ symmetry is broken.

It is useful (ref. 1, II-5-3) to define the effective thermal parameter T as

$$T = \frac{Kt + (N-K)t'}{N} \tag{30}$$

and the anisotropy g as

$$g = \frac{t' - t}{N} \tag{31}$$

The critical line where the $O(K)$ symmetry is broken is given by (ref. 1, II-5-3)

$$T = (\alpha g)^{1/\varphi} + \text{higher powers in } g \tag{32}$$

Near $t = t' = 0$, (32) can be rewritten as

$$t = t' - ct'^{\varphi} + \text{higher powers in } t' \tag{33}$$

with $c = N/\alpha$. The φ is called the crossover exponent, and the ε-expansion gives (ref. 1, II-5-3)

$$\varphi = 1 + \frac{N}{2(N+8)}\varepsilon + \frac{N^3 + 24N^2 + 68N}{4(N+8)^3}\varepsilon^2 + O(\varepsilon^3) > 1 \tag{34}$$

In this, $\varepsilon = 4 - d$, and d is the Euclidean dimension in which the model is defined. Let us consider the two-point function of the $\boldsymbol{\phi}_1$ field

$$G(r, t, t')\delta^{ij} = \langle \phi_1^i(0)\, \phi_1^j(r) \rangle \tag{35}$$

[6] The first K components of $\boldsymbol{\phi}$ form the vector $\boldsymbol{\phi}_1$ and the last $(N-K)$ components of $\boldsymbol{\phi}$ form the vector $\boldsymbol{\phi}_2$.

As before, we can write down the random walk representation of this two-point function. Instead of (25), we will now have

$$S(l, t, t') = e^{-tl} S_l(l, t, t') \tag{36}$$

where

$$S_l(l, t, t') = \frac{1}{Z(t, t')} \int [d\sigma] \\ \times \exp\left\{ -\frac{K}{2} \operatorname{Tr} \ln(H+t) - \frac{(N-K)}{2} \operatorname{Tr} \ln(H+t') - \frac{1}{2} \int d^3x\, \sigma^2 \right\} \\ \times \left[\int d^3r\, \Psi(l, r) \right] \tag{37}$$

and

$$Z(t, t') = \int [d\sigma] \\ \times \exp\left\{ -\frac{K}{2} \operatorname{Tr} \ln(H+t) - \frac{(N-K)}{2} \operatorname{Tr} \ln(H+t') - \frac{1}{2} \int d^3x\, \sigma^2 \right\} \tag{38}$$

(36) goes into (25) when $t = t'$. Further, the exponential factor e^{-tl} is the same in both (25) and (36) because in the latter case we are interested in the two-point function of the ϕ_1 field with the mass term being $(m_c^2 + t)(\phi_1 \cdot \phi_1)$ [see (29)]. The energy per unit length of the walk is

$$\theta(T, g) = t - f(t, t') \tag{39}$$

with

$$f(t, t') = \lim_{l \to \infty} \frac{\ln(S_l(l, t, t'))}{l} \tag{40}$$

The first term in (39) is from the exponential factor e^{-tl} in (36). Referring to (37), (38), and (40), we see that the t dependence in $f(t, t')$ vanishes when $K = 0$. But $\theta(T, g)$ has a dependence on t due to the first term in (39). This point is important and will be used in what follows. When $t = t'$, (37) goes to (26), (38) goes to (20), and (39) goes to (27).

To proceed further, we have to look at the scaling form of $\theta(T, g)$ near the critical line (32). From the discussion in ref. 1, II-5-3, it follows that θ has the form

$$\theta(T, g) = T^{\nu\theta} h_K(g/T^{\varphi}) \tag{41}$$

$h_K(x)$ is regular at $x = 0$, and the leading behavior in (41) is compatible with (6) when $g = 0$. (39) and (41) have to be consistent. For example, the calculations of

$$\left.\frac{\partial\theta(T, g)}{\partial g}\right|_{g=0, K=0} \tag{42}$$

from (39) and (41) must agree. This gives

$$-N = t^{\nu_\theta - \varphi} h_0'(0) \tag{43}$$

$h_K'(x)$ is the derivative of $h_K(x)$ with respect to x. The left-hand side of (43) is obtained by evaluating (42) using (39). We have used (30) and (31) and the fact that $f(t, t')$ in (39) does not depend on t when $K = 0$ [see (37) and (40)]. The right-hand side of (43) is obtained by evaluating (42) using (41). $h_0(x)$ is regular at $x = 0$ and $h_0'(0)$ is a constant. (43) is valid for a range of t near 0. This means that $h_0'(0) = -N$ and, more significantly, the relation of interest [cf. (8)], $\nu_\theta = \varphi$.

4. CONCLUSIONS

In this paper we analyzed the singular behavior of the random walk representation of the two-point function. The energy per unit length of the walk $\theta(t)$ is found to have a nontrivial dependence on the bare parameter t [cf. (6)]. This is attributed to the fact that the walk is taking place in the presence of background loops. For $O(N)$ models, the exponent ν_θ characterizing the nonanalytic behavior of $\theta(t)$ is shown to be the same as another exponent already known in the context of field theory, namely the crossover exponent φ [cf. (8)]. This connection enables us to derive a relation for the Hausdorff dimension of the walk in terms of standard exponents in field theory [cf. (9)].

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